Math 250 Test 1/Fall '13
Burger

100 ☺

Portfolio Grade 10

Na[redacted]

1. *(20pts)* Given that $2+i$ and -3 are roots of $p(x)$, factor the polynomial;

20

$$p(x) = 50x^5 - 115x^4 - 267x^3 + 1187x^2 - 1101x + 270$$

into irreducible polynomials in $\mathbb{Z}[x]$. $2+i$ $2-i$ -3

$(x-(2+i))(x-(2-i))$

$x^2 - x(2-i) - x(2+i) + (2+i)(2-i)$

$x^2 - 2x + \cancel{xi} - 2x - \cancel{xi} + 4 - \cancel{i^2}(-1)$

$x^2 - 4x + 5 \mid p(x)$

fact

$p(x) = (x^2-4x+5)(50x^3+85x^2-177x+54)$

$\rightarrow$ so $50x^3 + 85x^2 - 177x + 54 \mid p(x)$

factor

```
                (50)   (85)  (-177)  (54)
1 (-4) 5 |(50)(-115)(-267)(1187)(-1101)(270)
          50 (-200)(250)
              (85)(-517)(1187)
              (85)(-340)(425)
                   (-177)(762)(-1101)
                   (-177)(708)(-885)
                        (54)(-216)(270)
                        (54)(-216)(270)
                                     0
```

we know $x+3 \mid p(x)$ b/c -3 is root

```
          50 (-65) (18)
1 3 |(50)(85)(-177)(54)
     (50)(150)
        (-65)(-177)
        (-65)(-195)
              (18)(54)
              (18)(54)
                  0
```

$p(x) = (x^2-4x+5)(x+3)(50x^2-65x+18)$

need to factor

$$\boxed{p(x) = (x^2-4x+5)(x+3)(10x-9)(5x-2)}$$ ✓

factor $50x^2 - 65x + 18$

50 · 18
2 25 5 5
2 9 3 3

2 5 2
5 3 3

$50x^2 - 20x - 45x + 18$

$10x(5x-2) - 9(5x-2)$

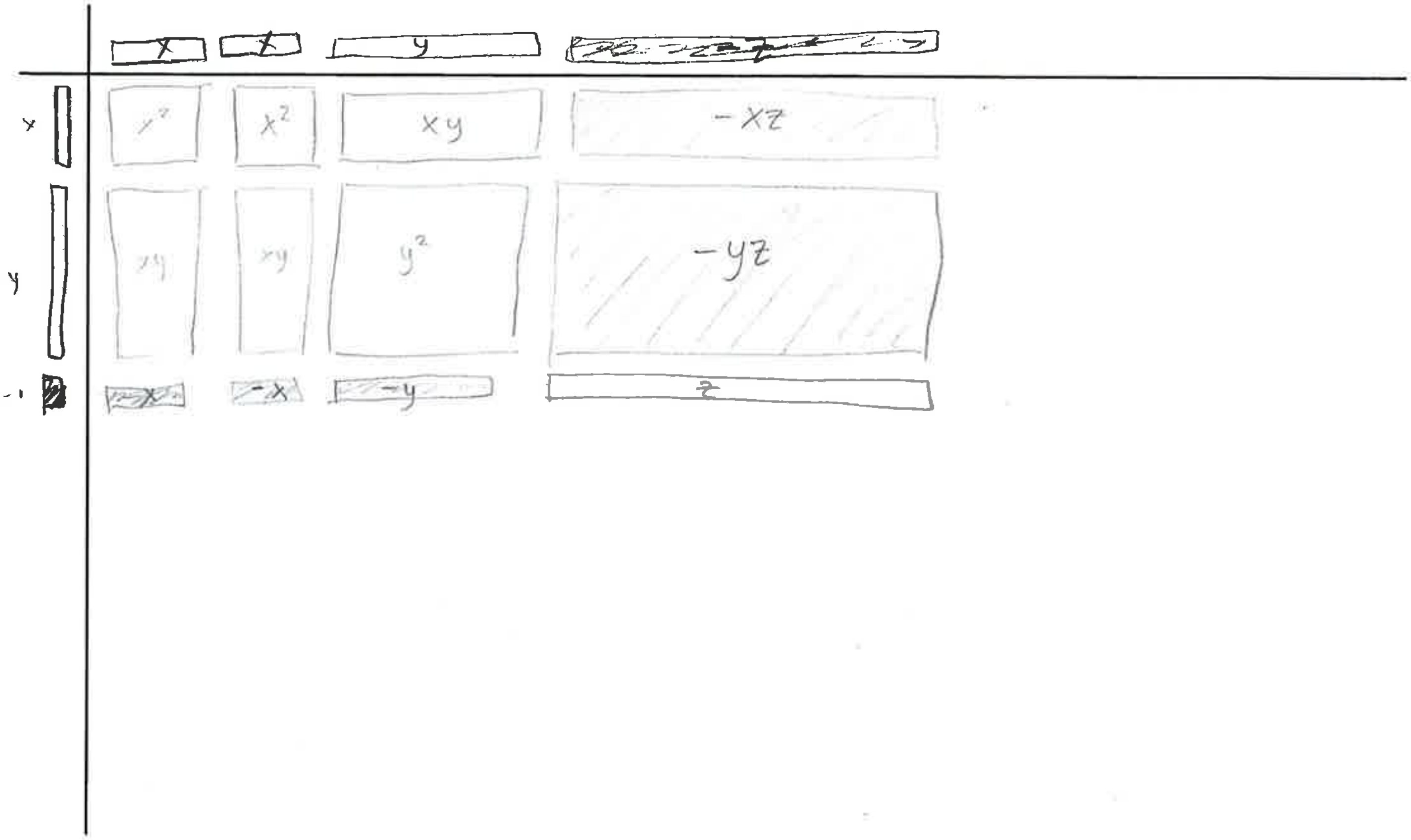
$(10x-9)(5x-2)$

Given: $x =$ $y =$ $z =$

2. *(10pts)* Draw a labeled *xyz-tile* diagram to solve:

$$\frac{2x^2+y^2+3xy-xz-yz-2x-y+z}{x+y-1} = 2x+y-z$$

10.



3. *(20pts)* In $\mathbb{Q}[x]$, factor $\overset{f(x)=}{x^8} + 3x^7 - 4x^4 - 12x^3 + 6x^2 + 20x + 6$ into a product of irreducible polynomials, justifying why your answer is composed of irreducible polynomials.

Rat Roots Thm

$\pm 6, \pm 3, \pm 2, \pm 1$

roots: -3 so $x+3 \mid f(x)$

$$\begin{array}{r|l} & 1\ 0\ 0\ 0\ (-4)\ 0\ 6\ 2 \\ 1\ 3 & 1\ 3\ 0\ 0\ (-4)(-12)\ 6\ (20)\ 6 \\ & 1\ 3 \\ & \quad 0\ 0\ 0\ (-4)(-12) \\ & \qquad (-4)(-12) \\ & \qquad\quad 0\ 6\ (20) \\ & \qquad\qquad 6\ (18) \\ & \qquad\qquad\quad 2\ 6 \\ & \qquad\qquad\quad 2\ 6 \\ & \qquad\qquad\qquad 0 \end{array}$$

20.

$f(x) = (x+3)\underbrace{(x^7 - 4x^3 + 6x + 2)}$

Let $p=2$: $2 \nmid 1$ $\quad 2 \mid -4$ $\quad 2 \mid 6$ $\quad 2^2 \nmid 2$

so by Eisenstein's Criterion, $x^7 - 4x^3 + 6x + 2$ is irreducible in $\mathbb{Q}[x]$ ✓

so $f(x) = (x+3)(x^7 - 4x^3 + 6x + 2)$ is a product of irreducible polynomials

15

4. *(15pts)* Use *Proposition 3* to demonstrate the irreducibility of the following polynomial, $p(x)$, over $\mathbb{Z}$:

$$p(x) = 7x^3 + 12x^2 - 2x + 14$$

$p(x)$ is irreducible in $\mathbb{Z}_{11}[x]$

so by Prop 3, $p(x)$ is irreducible in $\mathbb{Z}[x]$

$\mathbb{Z}_{11}[x]$ $\quad p(x) = 7x^3 + x^2 + 9x + 3$

$p(0) = 3 \neq 0$

$p(1) = 20 = 9 \bmod 11 \neq 0$

$p(2) = 81 = 4 \neq 0$

$p(3) = 228 = 8 \neq 0$

$p(4) = 503 = 8 \neq 0$

$p(5) = 948 = 2 \neq 0$

$p(6) = 1605 = 10 \neq 0$

$p(7) = 2516 = 8 \neq 0$

$p(8) = 3723 = 5 \neq 0$

$p(9) = 5268 = 10 \neq 0$

$p(10) = 7193 = 10 \neq 0$

$p(x)$ is irreducible in $\mathbb{Z}_{11}[x]$ because if $p(x)$ factored it would have a root in $\mathbb{Z}_{11}$.

$p(x)$ has no roots in $\mathbb{Z}_{11}$ $\therefore$ irreducible in $\mathbb{Z}_{11}[x]$

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

5. *(20pts)* In $\mathbb{Z}_3[x]$, given that $f(x) = 2x^4 + x^3 + x^2 + 2$ and $g(x) = 2x^3 + 2x^2 + 2$, then if possible, find $s(x), t(x) \in \mathbb{Z}_3[x]$, s.t.:

20

$$f(x) \cdot s(x) + g(x) \cdot t(x) = x^4 + 2x^3 + 2x + 1$$

11
2202 ⟌ 21102
2202
2112
2202
210

$f(x) = g(x)\cdot(x+1) + (2x^2+x)$

$(-x-1) = (2x+2)$

$\rightsquigarrow (2x^2+x) = f(x)\cdot(1) + g(x)\cdot(2x+2)$

12
210 ⟌ 2202
210
102
120
12

the gcd

$g(x) = (2x^2+x)(x+2) + (x+2)$

$(-x-2) = 2x+1)$

$\rightsquigarrow (x+2) = g(x)\cdot(1) + (2x^2+2)(2x+1)$

$x+2 = g(x)\cdot(1) + [f(x)\cdot(1) + g(x)\cdot(2x+2)](2x+1)$

$x+2 = f(x)\cdot(2x+1) + g(x)\cdot(x^2)$

$x^2+2x+4x+2 = x^2+2$
$+1$
$= x^2$

20
12 ⟌ 210
21
00

1002
12 ⟌ 12021
12
0021
21
0

$(x+2)(x^3+2) = f(x)\cdot(2x+1)(x^3+2) + g(x)(x^2)\cdot(x^3+2)$

$x^4+2x^3+2x+1 = f(x)\cdot(2x^4+x+x^3+2) + g(x)\cdot(x^5+2x^2)$

$= f(x)\cdot(2x^4+x^3+x+2) + g(x)\cdot(x^5+2x^2)$

$$\boxed{\text{So } x^4+2x^3+2x+1 = f(x)\cdot\underbrace{(2x^4+x^3+x+2)}_{s(x)} + g(x)\cdot\underbrace{(x^5+2x^2)}_{t(x)}}$$

$x^4+2x^3+2x+1 = (x+2)(x^3+2)$

gcd divides, so problem has solution

Check: $(2x^4+x^3+x^2+2)(2x^4+x^3+x+2) + (2x^3+2x^2+2)(x^5+2x^2)$

$\cancel{4x^8} + \cancel{2x^7} + \cancel{2x^5} + \cancel{4x^4} + \cancel{2x^7} + \cancel{x^6} + (x^4 + 2x^3 + \cancel{2x^6} + \cancel{x^5} + \cancel{x^3} + \cancel{2x^2} + \cancel{4x^4} + \cancel{2x^3} + 2x + 4$

$+ \cancel{2x^8} + \cancel{4x^5} + \cancel{2x^7} + \cancel{4x^4} + \cancel{2x^5} + \cancel{4x^2}$

$= x^4 + 2x^3 + 2x + 1$ ✓

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

6. *(15pts)* Given the polynomial $p(x) = 2x^4 + 2x^3 + x^2 + x + 2$, factor it 'completely' in $\mathbb{Z}_3[x]$ if reducible.

15.

$p(0) = 2 \neq 0$

$p(1) = 2 \neq 0$

$p(2) = 32 + 16 + 4 + 2 + 2 = 56 = 2 \neq 0$

so $p(x)$ has no roots in $\mathbb{Z}_3$

so if $p(x)$ factors it is an irreducible quadratic · irreducible quadratic of the form

$$(ax^2 + bx + c)(dx^2 + ex + f)$$

$$= adx^4 + aex^3 + afx^2 + bdx^3 + bex^2 + bfx + cdx^2 + cex + cf$$

$$= ad\,x^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

so $ad = 2$ and $cf = 2$, $ae + bd = 2$, $af + be + cd = 1$, $bf + ce = 1$

- let $a = 1, d = 2$ (this is the same as if $d = 2$ $a = 1$, the quadratics would just be in reverse order)

(i) suppose $c = 1, f = 2$. Then $af + be + cd = 1 \rightarrow 2 + be + 2 = 1$

$be = 0$

Then $e + 2b = 2$

Then $2b + e = 1$ > cannot both be true → contradiction

(ii) Suppose $c = 2, f = 1$ Then $ae + bd = 2 \rightarrow e + 2b = 2 \rightarrow 2 + 2 = 2$ contradiction

$bf + ce = 1 \rightarrow b + 2e = 1$ → $2 + 2 = 1$ ✓

$1 + 1 = 2$

$af + be + cd = 1 \rightarrow 1 + be + 4 = 1$

$be = 2$

let $b = 1, e = 2$

let $b = 2, e = 1$

so $a = 1, d = 2, c = 2, f = 1, b = 2, e = 1$ → $\boxed{p(x) = (x^2 + 2x + 2)(2x^2 + x + 1)}$

ok.